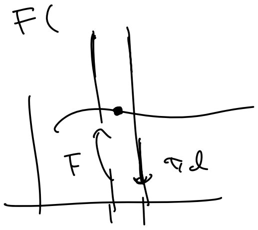

End of last class

$G : \mathbb{R}^n \longrightarrow \mathbb{R}^d \times \mathbb{R}^{n-d}$

(π_d, f)

F local inverse

locally, $S = f^{-1}(0) = F($



$G \circ F = id$

$\Rightarrow \pi_d \circ F|_{\mathbb{R}^d \times 0} = id_{\mathbb{R}^d \times 0}$

$F \circ G = id$

$\Rightarrow$ If $p \in f^{-1}(0)$,

$G(p) = (\pi_d(p), 0)$

$F \circ G(p) = F|_{\mathbb{R}^d \times 0} \circ \pi_d(p)$

$F|_{\mathbb{R}^d \times 0} \circ \pi_d = id|_S$

Note If f is a submersion, $T_p f^{-1}(0) = \ker(df_p)$

Thm: If $f : U^{\subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}^m$, $S^{m-r} \subseteq \mathbb{R}^m$ submfld, and $\operatorname{Im}(df_p) \oplus T_{f(p)} S = \mathbb{R}^m$, then $f^{-1}(S)$ is a $(n-r)$-submfld.

⌜
- WLOG, near p, $S = \mathbb{R}^{m-r} \subseteq \mathbb{R}^m$ (fund thm of imm)
- $S \xrightarrow{\pi_r} \mathbb{R}^r$ is a submersion at p ⌟

Dfn (Transversality)

- If $f : U \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^m$

  $f \pitchfork S$ if $\operatorname{Im}(df_p) \oplus T_{f(p)} S = \mathbb{R}^m \quad \forall p \in f^{-1}(S)$
- If $S, \hat{S} \subseteq \mathbb{R}^m$, $S \pitchfork \hat{S}$ if $T_p \hat{S} \oplus T_p S = \mathbb{R}^m \quad \forall p \in S \cap \hat{S}$
- If $f, g : U \to \mathbb{R}^m$, $f \pitchfork g$ if $\forall p \in U$ s.t. $f(p) = g(p)$,

  $df_p - dg_p : T_p U \twoheadrightarrow \mathbb{R}^m$

Application

- $f \pitchfork S \Rightarrow f^{-1}(S)$ submfld
- $S \pitchfork \hat{S} \Rightarrow S \cap \hat{S}$ submfld (apply to $f = \iota : \hat{S} \hookrightarrow \mathbb{R}^n$)
- $f \pitchfork g \Rightarrow \{f = g\}$ submfld (dim $n-m$)

$$\begin{tikzcd} U^{\subseteq \mathbb{R}^n} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R}^m \end{tikzcd}$$

$\vdash$ $(f,g) \pitchfork \Delta$ where $(f,g) : U \to \mathbb{R}^{2m}$, $\Delta \subseteq \mathbb{R}^{2m}$ is the diagonal $\lrcorner$

Rmk Everything works same with $\mathbb{R}^n$ replaced by an n-dim'l mfld, eg.

Thm (IFT for submanifolds)
If S, $\hat{S}$ submflds, $df_p : T_pS \longrightarrow T_{f(p)}\hat{S}$ is an isomorphism, then f is a local diffeomorphism.

$\vdash$
$$\begin{tikzcd} S \arrow[r, "f"] & \hat{S} \\ U \arrow[u, "\psi"] \arrow[r, shift left] & \hat{U} \arrow[u, "\hat{\psi}"'] \arrow[l, dashed, shift left, "IFT"] \end{tikzcd}$$
$\lrcorner$

Today we describe a different way to build new mflds, and see the role of abstract manifolds.

<u>Note</u> Every construction last class (limit-type constructions), when applied to submflds of $\mathbb{R}^n$, gave more sub-mflds of $\mathbb{R}^n$.

<u>Today</u> colimit-type constructions

<u>Recall</u> If $S_1 \subseteq \mathbb{R}^{n_1}$, $S_2 \subseteq \mathbb{R}^{n_2}$ d-mflds, then $S_1 \sqcup S_2$ is an <u>abstract</u> d-mfld, but doesn't come w/ canonical embedding.

What about ∞ many?

$\{S_\alpha \subseteq \mathbb{R}^{n_\alpha}\}_{\alpha \in A}$ d-manifolds, then $\bigsqcup_\alpha S_\alpha$ is an <u>abstract</u> manifold iff A countable

$\boxed{\text{embeds}}$

<u>Thm</u> If A countable, then $\bigsqcup_\alpha S_\alpha \hookrightarrow \mathbb{R}^N$ for some N

<u>Pf</u> Note if $n_\alpha \leq N-1$ $\forall \alpha$, then $\bigsqcup S_\alpha \hookrightarrow \mathbb{R}^N \times \mathbb{Z} \hookrightarrow \mathbb{R}^{N+1}$

Sard/Whitney: If $S^d \subseteq \mathbb{R}^n$ and $n > 2d+1$, you can always project S into some $\mathbb{R}^{n-1}$ and get an embedding.

A little category theory:

<u>product</u> X, Y finite sets, then $X \times Y$ is a set such that

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

and if
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$
then $\exists !$
$$\begin{tikzcd} Z \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times Y \arrow[r] \arrow[d] & Y \\ & X & \end{tikzcd}$$

making the diagram commute.

Q: is $X \times Y$ unique?

A: yes if ∃ exists in the following sense: if A and B are two sets satisfying product condition, then $\exists !$ maps

$$\begin{tikzcd} A \arrow[r, bend left] & B \arrow[l, bend left] \end{tikzcd}$$

making diagram commute

inverses? <u>yes</u> b/c ∃ map $A \circlearrowright$ making diagram commute.

Q: does $X \times Y$ exist?

A: <u>yes</u> if $X = (x_1 — x_n)$, $Y = (y_1 --, y_m)$ then

$X \times Y = (z_{11}, z_{12} — z_{1n}, z_{21} - z_{2n}, — . z_{m1}, — z_{mn})$

$$\begin{tikzcd} z_{ij} \arrow[r] \arrow[d] & y_j \\ x_i & \end{tikzcd}$$

Similarly, if $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, the equalizer of f and g is

$\{f=g\} \xrightarrow{\iota} X$ s.t. $f \circ \iota = g \circ \iota$ and $\forall\ Z \xrightarrow{j} X$ s.t.

$\dots$, $\exists !\ Z \longrightarrow$

Same $\exists$ and ! discussion applies.

If $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ mflds then

- products are mflds
- equalizers are mflds if $f \pitchfork g$

and they all are straightforwardly ! b/c they are naturally submflds of Euclidean space.

Colimits reverse arrows:

Sets

coproduct

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drr, bend right] & X \sqcup Y \arrow[dr] & \\
 & & Z
\end{tikzcd}$$

disjoint union

Recall If $S \subseteq \mathbb{R}^n$, $\hat{S} \subseteq \mathbb{R}^m$ are. d-mflds, then it is always possible to fit $S \sqcup \hat{S}$ inside $\mathbb{R}_{\max(n,m)}$, but maybe not canonically.

Nonetheless, $S \sqcup \hat{S}$ is unique in the sense above by general category theory!

coequalizers

$$\begin{tikzcd}
R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[rr, bend left] \arrow[rr, bend right] & X \arrow[r, "\pi"] & X/R
\end{tikzcd}$$

What is this??

Well, $(f,g) : R \to X \times X$, and X/R depends only on its image;

Suggestively, write $x_1 \sim x_2$ for $(x_1, x_2) \in \text{image}(R)$.

" . $\pi(x_1) = \pi(x_2)$

if $x_1 \sim x_2$ then ....

Dfn R is an equiv. relation if $x \sim x$ $\forall x$ and $x \sim y$, $y \sim z \Rightarrow x \sim z$.

Then X/R is the quotient of X by the equiv relation generated by Image(R).

② Abstract manifolds

- If $S \overset{\text{subset}}{\subseteq} \mathbb{R}^n$, we know a priori what it means for $U^{\subseteq \mathbb{R}^m} \xrightarrow{s} S$, $S' \xrightarrow{F} U^{\subseteq \mathbb{R}^m}$ to be smooth, so you can ask is an atlas smooth?, any 2 smooth atlases are equivalent.
- On an abstract manifold, the atlas defines what it means for $U \to M$, $M \to U$ to be smooth, i.e. if they're locally smooth in charts. You can have different atlases on the same topological space.

Summary

There is an equivalence of categories

$$\left\{ \begin{array}{c} \text{Submanifolds} \\ \text{of } \mathbb{R}^N, \\ \text{diffeo} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Abstract} \\ \text{mflds,} \\ \text{diffeo} \end{array} \right\}$$